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ANALYSIS OF THE STEADY OSCILLATIONS OF A PLANE ABSOLUTELY RIGID INCLUSION IN A THREE-DIMENSIONAL ELASTIC BODY BY THE BOUNDARY ELEMENT METHOD†

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The three-dimensional problem of the involvement of a plane absolutely rigid inclusion of specified mass in the field of a harmonic wave propagating in an infinite elastic body is considered by means of integral equations with a singularity of the Helmholtz potential. A boundary-element algorithm is proposed for constructing a discrete analogue of the equations, taking into account the fact that the solutions belong to a class of functions which increase on the contour of the integration region (the region of the defect). The dependence of the displacements of the inclusions as a rigid whole and also the stress concentration in its neighbourhood on the wave number is investigated for two cases of the diffraction by a disc-shaped inclusion of a plane longitudinal wave with a wave front that is parallel and perpendicular to it. © 2003 Elsevier Science Ltd. All rights reserved.

In addition to cracks, thin foreign inclusions are objects around which stress concentrations occur, for which the ideas of their stress intensity factors have been introduced [1], which are important from the point of view of fracture mechanics. However, although the dynamic problems of the crack theory have been considered in numerous publications [2–6], inertial effects in elastic bodies with thin inclusions have been investigated to a much lesser extent and are concerned with two-dimensional formulations of the problems for tunnel defects [7, 8]. Only scalar problems of the dynamic interaction of an absolutely rigid disc with an acoustic medium have been considered in a three-dimensional formulation [9].

1. REDUCTION OF THE PROBLEM TO BOUNDARY INTEGRAL EQUATIONS

Suppose an infinite isotropic elastic body (a matrix) contains a plane absolutely rigid inclusion of mass M, which occupies a region S in the x_1Ox_2 plane. A system of coordinates $Ox_1x_2x_3$ is introduced in such a way that the centre O coincides with the centre of mass of the defect, while the values $x_3 = \pm 0$ correspond to the demarcation surfaces $S^{\pm}$ of the inclusion and the body. A stress–strain state is excited by an incident harmonic wave with a specified distribution of the displacements $\mathbf{U}^{in}(\mathbf{x}^*, t) = \mathbf{u}^{in}(\mathbf{x}^*)\exp(-i\omega t)$ in the space $\mathbf{x}^*(x_1, x_2, x_3)$ and in time t, where $\mathbf{u}^{in}(u_1^{in}, u_2^{in}, u_3^{in})$ is their amplitude and ω is the cyclic frequency of the oscillations.

Starting from the superposition principle and omitting the exponential time factor, which is common for quantities of the steady process, the complete diffraction field of the displacements $\mathbf{u}^D$ in the body with the inclusion can be represented in the form

$$\mathbf{u}^D(\mathbf{x}^*) = \mathbf{u}^{in}(\mathbf{x}^*) + \mathbf{u}(\mathbf{x}^*) \tag{1.1}$$

Here $\mathbf{u}(u_1, u_2, u_3)$ are the unknown displacements of the waves reflected from the inclusion, which satisfy the radiation conditions at infinity. The key equation for the components of relation (1.1) is the Lamé equation of harmonic oscillations [10]

$$\omega_1^{-2}\, \nabla\, (\nabla \cdot \mathbf{u}) - \omega_1^{-2}\, \nabla \times (\nabla \times \mathbf{u}) + \mathbf{u} = 0 \tag{1.2}$$

where ∇ is the three-dimensional Nabla vector, $\omega_j = \omega/c_j$ ($j = 1, 2$) are wave numbers, and c_1 and c_2 are the propagation velocities of longitudinal and transverse waves in the body.

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The properties of the inclusion as a rigid body, the motion of which can only be translational with a small displacement $\mathbf{u}^0(u_1^0, u_2^0, u_3^0)$ of its centre of mass and rotational around the coordinate axes with a small angle of rotation $\mathbf{\Omega}(\Omega_1, \Omega_2, \Omega_3)$, are modelled by the boundary conditions. Hence, we obtain the conditions of continuity of the displacements in the region S, where, component-by-component

$$u_j(\mathbf{x}) = -u_j^{in}(\mathbf{x}) + u_j^0 - \Omega_3 x_{3-j}, \quad j = 1,2; \quad u_3(\mathbf{x}) = -u_3^{in}(\mathbf{x}) + u_3^0 - \Omega_2 x_1 + \Omega_1 x_2 \tag{1.3}$$
$$\mathbf{x}(x_1, x_2) = \mathbf{x}^*(x_1, x_2, \pm 0) \in S$$

The response of the body to the presence in it of an inclusion is defined by the principal vector of the forces acting on the defect with projections $P_j (j = 1, 2, 3)$ and the moments of these forces $Z_j (j = 1, 2, 3)$ about the coordinate axes. Additional relations, which connect the quantities P_j, Z_j and u_j^0, Ω follow from the well-known equations of the steady oscillations of the inclusion (i_j is the radius of inertia of the inclusion about the Ox_j axis)

$$P_j + M\omega^2 u_j^0 = 0, \quad Z_j + M\omega^2 i_j^2 \Omega_j = 0; \quad j = 1,2,3 \tag{1.4}$$

In order to give an integral representation of the solution of problem (1.2)–(1.4) we will use Somigliana's formula in the form [11]

$$u_j(\mathbf{x}^*) = \sum_{i=1}^{3} \iint_{S^+ \cup S^-} [p_i(\boldsymbol{\xi}) U_{ij}(\boldsymbol{\xi} - \mathbf{x}^*) - u_i(\boldsymbol{\xi}) T_{ij}(\boldsymbol{\xi} - \mathbf{x}^*)] dS_{\xi}, \quad j = 1,2,3 \tag{1.5}$$

where $U_{ij} (i, j = 1, 2, 3)$ are the elements of the matrix of the fundamental solutions of the steady dynamic problem of the theory of elasticity in displacements [10], T_{ij} $(i, j = 1, 2, 3)$ are the forces on the areas with normals to the surfaces $S^{\pm}$, corresponding to these solutions, and p_i $(i = 1, 2, 3)$ are the projections of the unknown forces acting on the body from the side of the inclusion.

The following equalities for the fundamental solutions are obvious

$$U_{ij}(\boldsymbol{\xi} - \mathbf{x}^*)\big|_{\xi \in S^+} = U_{ij}(\boldsymbol{\xi} - \mathbf{x}^*)\big|_{\xi \in S^-}, \quad T_{ij}(\boldsymbol{\xi} - \mathbf{x}^*)\big|_{\xi \in S^+} = -T_{ij}(\boldsymbol{\xi} - \mathbf{x}^*)\big|_{\xi \in S^-} \tag{1.6}$$

Then, by introducing notation for the jumps in the stress components σ_{i3} at the point where the defect is located

$$\Delta\sigma_i(\mathbf{x}) = -\frac{1}{4\pi}[\sigma_{i3}^+ - \sigma_{i3}^-] = \frac{1}{4\pi}\left[p_i\big|_{S^+} + p_i\big|_{S^-}\right], \quad j = 1,2,3, \quad \mathbf{x}(x_1, x_2) \in S \tag{1.7}$$
$$\sigma_{i3}^{\pm}(\mathbf{x}) = \lim_{x_3 \to \pm 0} \sigma_{i3}(\mathbf{x}^*)$$

relations (1.5) are converted to the following form (everywhere henceforth, unless otherwise stated, the integration is carried out over the region S)

$$u_j(\mathbf{x}^*) = 4\pi \sum_{i=1}^{3} \iint \Delta\sigma_i(\boldsymbol{\xi}) U_{ij}(\boldsymbol{\xi} - \mathbf{x}^*) dS_{\xi}, \quad j = 1,2,3 \tag{1.8}$$

or, taking into account the explicit expressions for the functions U_{ij}

$$u_j(\mathbf{x}^*) = \frac{1}{G} \iint \left\{ \Delta\sigma_j(\boldsymbol{\xi}) \frac{\exp(i\omega_2 |\mathbf{x}^* - \boldsymbol{\xi}|)}{|\mathbf{x}^* - \boldsymbol{\xi}|} - \frac{1}{\omega_2^2} \frac{\partial}{\partial x_j} \left[\Delta\sigma_1(\boldsymbol{\xi}) \frac{\partial}{\partial x_1} + \Delta\sigma_2(\boldsymbol{\xi}) \frac{\partial}{\partial x_2} + \right.\right. \tag{1.9}$$
$$\left.\left. + \Delta\sigma_3(\boldsymbol{\xi}) \frac{\partial}{\partial x_3} \right] \left[\frac{\exp(i\omega_1 |\mathbf{x}^* - \boldsymbol{\xi}|)}{|\mathbf{x}^* - \boldsymbol{\xi}|} - \frac{\exp(i\omega_2 |\mathbf{x}^* - \boldsymbol{\xi}|)}{|\mathbf{x}^* - \boldsymbol{\xi}|} \right] \right\} dS_{\xi}, \quad j = 1,2,3$$

Here G is the shear modulus and $|\mathbf{x}^* - \xi|$ is the distance between the point $\mathbf{x}^*(x_1, x_2, x_3)$ and the point of integration $\xi(\xi_1, \xi_2)$.

Hence, the displacements at an arbitrary point of the body with the inclusion can be represented by the convolutions (1.9) of the jumps in the stresses in the area of the defect with regular kernels of the

Helmholtz potentials. Using relations (1.4) and (1.7) in terms of the functions $\Delta\sigma_i$ (i = 1, 2, 3) we can also write the displacements and the rotations of the inclusion as a rigid whole, namely

$$\begin{aligned} u_j^0 &= \frac{4\pi}{\omega^2 M}\iint \Delta\sigma_j(\boldsymbol{\xi})dS_\xi, \quad j=1,2,3 \\ \Omega_j &= \frac{(-1)^{j+1}4\pi}{\omega^2 M i_j^2}\iint \xi_{3-j}\Delta\sigma_3(\boldsymbol{\xi})dS_\xi, \quad j=1,2 \\ \Omega_3 &= -\frac{4\pi}{\omega^2 M i_3^2}\iint[\xi_2\Delta\sigma_1(\boldsymbol{\xi})-\xi_1\Delta\sigma_2(\boldsymbol{\xi})]dS_\xi \end{aligned} \tag{1.10}$$

To determine the jumps in the stresses $\Delta\sigma_i$ (i = 1, 2, 3) we will use conditions (1.3). By satisfying these conditions using representations (1.9) and (1.10), we obtain the following system of boundary integral equations of the Helmholtz potential type in terms of the functions $\Delta\sigma_i$

$$\begin{aligned} &\iint \Delta\sigma_3(\boldsymbol{\xi})R_3(\mathbf{x},\boldsymbol{\xi})dS_\xi = -\omega_2^2 G u_3^{in}(\mathbf{x}), \quad \mathbf{x}(x_1,x_2)\in S \\ &\iint[\Delta\sigma_j(\boldsymbol{\xi})R_j(\mathbf{x},\boldsymbol{\xi})+\Delta\sigma_{3-j}(\boldsymbol{\xi})R_{j(3-j)}(\mathbf{x},\boldsymbol{\xi})]dS_\xi = -\omega_2^2 G u_j^{in}(\mathbf{x}), \quad j=1,2, \quad \mathbf{x}(x_1,x_2)\in S \end{aligned} \tag{1.11}$$

The kernels R_j (j = 1, 2, 3), R_{12} and R_{21}, after carrying out the differentiation operations, take the form

$$\begin{aligned} R_j(\mathbf{x},\boldsymbol{\xi}) &= L_1(|\mathbf{x}-\boldsymbol{\xi}|)-\frac{(x_j-\xi_j)^2}{|\mathbf{x}-\boldsymbol{\xi}|^2}L_2(|\mathbf{x}-\boldsymbol{\xi}|)-\frac{4\pi G}{c_2^2 M}\left(1+\frac{\xi_{3-j}x_{3-j}}{i_3^2}\right), \quad j=1,2 \\ R_3(\mathbf{x},\boldsymbol{\xi}) &= L_1(|\mathbf{x}-\boldsymbol{\xi}|)-\frac{4\pi G}{c_2^2 M}\left(1+\frac{\xi_1 x_1}{i_2^2}+\frac{\xi_2 x_2}{i_1^2}\right) \\ R_{ij}(\mathbf{x},\boldsymbol{\xi}) &= -\frac{(x_1-\xi_1)(x_2-\xi_2)}{|\mathbf{x}-\boldsymbol{\xi}|^2}L_2(|\mathbf{x}-\boldsymbol{\xi}|)+\frac{4\pi G}{c_2^2 M}\frac{\xi_i x_j}{i_3^2}, \quad i,j=1,2, \quad i\neq j \\ L_i(r) &= l_{i1}(r)\frac{\exp(i\omega_1 r)}{r^3}-l_{i2}(r)\frac{\exp(i\omega_2 r)}{r^3}, \quad i=1,2 \\ l_{11}(r) &= 1-i\omega_1 r, \quad l_{12}(r)=1-i\omega_2 r-\omega_2^2 r^2, \quad l_{2j}(r)=3-3i\omega_j r-\omega_j^2 r^2, \quad j=1,2 \end{aligned} \tag{1.12}$$

The first equation of system (1.11) corresponds to the problem of the transverse oscillations of the inclusion in an elastic body (from this we can determine the function $\Delta\sigma_3$, and then from formulae (1.10) we can determine the displacement u_3^0 and the rotations Ω_1 and Ω_2). The system of two remaining equations corresponds to the problem of the longitudinal oscillations of the inclusion (from these we can determine the functions $\Delta\sigma_1$ and $\Delta\sigma_2$ and then the displacements u_1^0 and u_2^0 and the rotation Ω_3). The kernels of the boundary integral equation contain a polar singularity, which can be separated using integrals of the Newton (static) potential type by an identical transformation of system (1.11) to the form

$$\begin{aligned} &A\iint \frac{\Delta\sigma_3(\boldsymbol{\xi})}{|\mathbf{x}-\boldsymbol{\xi}|}dS_\xi+\iint \Delta\sigma_3(\boldsymbol{\xi})\left[\frac{1}{\omega_2^2}R_3(\mathbf{x},\boldsymbol{\xi})-\frac{A}{|\mathbf{x}-\boldsymbol{\xi}|}\right]dS_\xi = -Gu_3^{in}(\mathbf{x}), \quad \mathbf{x}(x_1,x_2)\in S \\ &\iint \frac{\Delta\sigma_j(\boldsymbol{\xi})}{|\mathbf{x}-\boldsymbol{\xi}|}\left[A-B\frac{(x_j-\xi_j)^2}{|\mathbf{x}-\boldsymbol{\xi}|^2}\right]dS_\xi - B\iint \Delta\sigma_{3-j}(\boldsymbol{\xi})\frac{(x_1-\xi_1)(x_2-\xi_2)}{|\mathbf{x}-\boldsymbol{\xi}|^3}dS_\xi + \\ &+\iint\left\{\Delta\sigma_j(\boldsymbol{\xi})\left[\frac{1}{\omega_2^2}R_j(\mathbf{x},\boldsymbol{\xi})-\frac{A}{|\mathbf{x}-\boldsymbol{\xi}|}+B\frac{(x_j-\xi_j)^2}{|\mathbf{x}-\boldsymbol{\xi}|^3}\right]+\Delta\sigma_{3-j}(\boldsymbol{\xi})\left[\frac{1}{\omega_2^2}R_{j(3-j)}(\mathbf{x},\boldsymbol{\xi})+\right.\right. \\ &\left.\left.+B\frac{(x_1-\xi_1)(x_2-\xi_2)}{|\mathbf{x}-\boldsymbol{\xi}|^3}\right]\right\}dS_\xi = -Gu_j^{in}(\mathbf{x}), \quad j=1,2, \quad \mathbf{x}(x_1,x_2)\in S \end{aligned} \tag{1.13}$$

where

$$A=\frac{\gamma^2+1}{2},\quad B=\frac{\gamma^2-1}{2},\quad \gamma^2=\frac{c_2^2}{c_1^2}=\frac{1-2\nu}{2(1-\nu)}$$

and ν is Poisson's ratio. The last integrals on the left-hand sides of Eqs (1.13) are regular, which can be shown by expanding the exponential functions occurring in the kernels R_j and R_{ij} in series in the quantities $|\mathbf{x}-\boldsymbol{\xi}|$. Note that the characteristic part of (1.13) forms an integral operator of the equations of the three-dimensional static problem of a laminate with an absolutely rigid inclusion in an infinite elastic body [12].

2. THE CONSTRUCTION OF A DISCRETE ANALOGUE OF THE BOUNDARY INTEGRAL EQUATION

According to well-known results [12, 13], in the case of a disc-shaped inclusion of radius a, the continuity of the displacements in the neighbourhood of its edge will be ensured if the function $\Delta\sigma_j$ is represented as

$$\Delta\sigma_j(\mathbf{x})=\frac{\alpha_j(\mathbf{x})}{\sqrt{a^2-x_1^2-x_2^2}},\quad j=1,2,3,\quad \mathbf{x}(x_1,x_2)\in S \tag{2.1}$$

where $\alpha_j(\mathbf{x})$ ($j = 1, 2, 3$) are unknown functions. Substituting expressions (2.1) into relation (1.11) or (1.13), we obtain integral equations which, in addition to a polar singularity at the point $\boldsymbol{\xi} = \mathbf{x}$, has a root singularity on the contour of the region of integration. To avoid having to change to a polar system of coordinates, we will first use the following interpretation of the particular integrals in the region S

$$\begin{aligned}
&\iint\alpha(\boldsymbol{\xi})\frac{dS_{\xi}}{\Delta_1(\mathbf{x},\boldsymbol{\xi})}=\pi^2\alpha(\mathbf{x})+\iint[\alpha(\boldsymbol{\xi})-\alpha(\mathbf{x})]\frac{dS_{\xi}}{\Delta_1(\mathbf{x},\boldsymbol{\xi})}\\
&\iint\alpha(\boldsymbol{\xi})(x_j-\xi_j)^2\frac{dS_{\xi}}{\Delta_3(\mathbf{x},\boldsymbol{\xi})}=\frac{\pi^2}{2}\alpha(\mathbf{x})+\iint[\alpha(\boldsymbol{\xi})-\alpha(\mathbf{x})](x_j-\xi_j)^2\frac{dS_{\xi}}{\Delta_3(\mathbf{x},\boldsymbol{\xi})},\quad j=1,2\\
&\iint\alpha(\boldsymbol{\xi})(x_1-\xi_1)(x_2-\xi_2)\frac{dS_{\xi}}{\Delta_3(\mathbf{x},\boldsymbol{\xi})}=\iint[\alpha(\boldsymbol{\xi})-\alpha(\mathbf{x})](x_1-\xi_1)(x_2-\xi_2)\frac{dS_{\xi}}{\Delta_3(\mathbf{x},\boldsymbol{\xi})}\\
&\Delta_i(\mathbf{x},\boldsymbol{\xi})=\sqrt{a^2-\xi_1^2-\xi_2^2}\,|\mathbf{x}-\boldsymbol{\xi}|^i,\quad i=1,3
\end{aligned} \tag{2.2}$$

Here we have taken into account the exact values of the integrals

$$\iint\frac{dS_{\xi}}{\Delta_1(\mathbf{x},\boldsymbol{\xi})}=\pi^2,\quad \iint(x_1-\xi_1)^i(x_2-\xi_2)^{2-i}\frac{dS_{\xi}}{\Delta_3(\mathbf{x},\boldsymbol{\xi})}=\begin{cases}\pi^2/2, & \text{if } i=0,2\\ 0, & \text{if } i=1\end{cases};\quad \mathbf{x}\in S \tag{2.3}$$

It follows from obvious considerations that the integrands on the right-hand sides of (2.2) are bounded at the point $\boldsymbol{\xi} = \mathbf{x}$, and hence numerical integration in the corresponding integrals was carried out along the region S^0, which is formed from S by removing a small neighbourhood of this point.

The next step in regularizing the boundary integral equations consists of changing the variables

$$\begin{aligned}
&x_1=a\sin y_1\cos y_2,\quad x_2=a\sin y_1\sin y_2\\
&\xi_1=a\sin\eta_1\cos\eta_2,\quad \xi_2=a\sin\eta_1\sin\eta_2
\end{aligned} \tag{2.4}$$

where $\mathbf{y}(y_1, y_2)$ and $\boldsymbol{\eta}(\eta_1, \eta_2)$ are new variables, which vary in the rectangular region $\tilde{S}\{0 \leqslant y_1, \eta_1 \leqslant \pi/2; 0 \leqslant y_2, \eta_2 \leqslant 2\pi\}$. By making the replacement (2.4) we eliminate the singularity on the contour of the region of integration, when $\eta_1 = \pi/2$. By combining relations (2.1), (2.2) and (2.4), we obtain a regular representation of boundary integral equations (1.11) (or (1.13)) in the form

$$A\tilde{\alpha}_3(\mathbf{y})\left[\pi^2-\iint_{\tilde{S}^0}\frac{\sin\eta_1}{R(\mathbf{y},\boldsymbol{\eta})}dS_{\eta}\right]+\frac{1}{\kappa^2}\iint_{\tilde{S}^0}\tilde{\alpha}_3(\boldsymbol{\eta})\tilde{R}_3(\mathbf{y},\boldsymbol{\eta})\sin\eta_1 dS_{\eta}=-G\tilde{u}_3^{in}(\mathbf{y}),\quad \mathbf{y}(y_1,y_2)\in\tilde{S}$$

$$\tilde{\alpha}_j(\mathbf{y})\left\{A\left[\pi^2 - \iint\limits_{\tilde{S}^0} \frac{\sin\eta_1}{R(\mathbf{y},\boldsymbol{\eta})} dS_\eta\right] - B\left[\frac{\pi^2}{2} - \iint\limits_{\tilde{S}^0} \Phi_j(\mathbf{y},\boldsymbol{\eta})\sin\eta_1 dS_\eta\right]\right\} +$$
$$+ B\tilde{\alpha}_{3-j}(\mathbf{y})\iint\limits_{\tilde{S}^0} \Psi(\mathbf{y},\boldsymbol{\eta})\sin\eta_1 dS_\eta + \frac{1}{\kappa^2}\iint\limits_{\tilde{S}^0} [\tilde{\alpha}_j(\boldsymbol{\eta})\tilde{R}_j(\mathbf{y},\boldsymbol{\eta}) + \tag{2.5}$$
$$+ \tilde{\alpha}_{3-j}(\boldsymbol{\eta})\tilde{R}_{j(3-j)}(\mathbf{y},\boldsymbol{\eta})]\sin\eta_1 dS_\eta = -G\tilde{u}_j^{in}(\mathbf{y}), \quad j=1,2, \quad \mathbf{y}(y_1,y_2)\in\tilde{S}$$

Here

$$R(\mathbf{y},\boldsymbol{\eta}) = [\sin^2 y_1 + \sin^2\eta_1 - 2\sin y_1 \sin\eta_1 \cos(y_2-\eta_2)]^{1/2}$$
$$\Phi_j(\mathbf{y},\boldsymbol{\eta}) = \frac{1}{R^3(\mathbf{y},\boldsymbol{\eta})}[\delta_{1j}(\sin y_1\cos y_2 - \sin\eta_1\cos\eta_2)^2 + \delta_{2j}(\sin y_1 \sin y_2 - \sin\eta_1\sin\eta_2)^2]$$
$$\Psi(\mathbf{y},\boldsymbol{\eta}) = \frac{1}{R^3(\mathbf{y},\boldsymbol{\eta})}(\sin y_1\cos y_2 - \sin\eta_1\cos\eta_2)(\sin y_1\sin y_2 - \sin\eta_1\sin\eta_2)$$

δ_{ij} is the Kronecker delta, $\kappa = \omega_2 a$ is the normalized wave number, $\tilde{S}^0$ is the representation of the region S^0 when replacement (2.4) is made (in the region $\tilde{S}^0$ the points y and η do not coincide), and $\tilde{\alpha}_j$, $\tilde{u}_j^{in}$, $\tilde{R}_j$ (j = 1, 2, 3), $\tilde{R}_{12}$, $\tilde{R}_{12}$ are complex functions, introduced as

$$\tilde{\alpha}_j(\mathbf{y}) = \alpha_j(\hat{\mathbf{x}}), \quad \tilde{u}_j^{in}(\mathbf{y}) = u_j^{in}(\hat{\mathbf{x}}) \tag{2.6}$$
$$\tilde{R}_j(\mathbf{y},\boldsymbol{\eta}) = a^3 R_j(\hat{\mathbf{x}},\hat{\boldsymbol{\xi}}), \quad \tilde{R}_{ij}(\mathbf{y},\boldsymbol{\eta}) = a^3 R_{ij}(\hat{\mathbf{x}},\hat{\boldsymbol{\xi}})$$
$$\hat{\mathbf{x}} = \mathbf{x}(a\sin y_1\cos y_2, a\sin y_1 \sin y_2), \quad \hat{\boldsymbol{\xi}} = \boldsymbol{\xi}(a\sin\eta_1\cos\eta_2, a\sin\eta_1\sin\eta_2)$$

The discretization of Eqs (2.5) is based on uniform subdivision of the region $\tilde{S}$ by a grid Q of rectangular elements $\tilde{S}_q$, where q = 1, ..., Q, $\tilde{S} = \bigcup_{q=1}^{Q} \tilde{S}_q$, $\tilde{S}_q \cap \tilde{S}_p = \emptyset$, $q \neq p$ and the approximation of the required functions $\tilde{\alpha}_j$ (j = 1, 2, 3) by interpolation polynomials

$$\tilde{\alpha}_j(\mathbf{y}) = \sum_{q=1}^{Q} \tilde{\alpha}_{jq}\theta_q(\mathbf{y}), \quad j=1,2,3 \tag{2.7}$$

Here $\tilde{\alpha}_{jq} = \tilde{\alpha}_j(\mathbf{y}_q)$ is the value of the required function at the nodal point $\mathbf{y}_q(y_{1q}, y_{2q})$ in the middle of the qth element and θ_q (q = 1, ..., Q) are weighting functions with the properties $\theta_q(\mathbf{y}_p) = \delta_{qp}$.

Using a collocation scheme to satisfy Eqs (2.5), we arrive at the following systems of linear algebraic equations of dimension $Q \times Q$ for the problem of the transverse oscillations of an inclusion, and of dimension $2Q \times 2Q$ for the problem of the longitudinal oscillations of an inclusion with respect to the quantities $\tilde{\alpha}_{jq}$

$$\sum_{q=1}^{Q} h_{3iq}\tilde{\alpha}_{3q} = -G\tilde{u}_3^{in}(\mathbf{y}_i), \quad i=1,\dots,Q \tag{2.8}$$
$$\sum_{q=1}^{Q} [h_{jiq}\tilde{\alpha}_{jq} + h_{j(3-j)iq}\tilde{\alpha}_{(3-j)q}] = -G\tilde{u}_j^{in}(\mathbf{y}_i), \quad i=1,\dots,Q, \quad j=1,2$$

The coefficients h_{jiq}, $h_{j(3-j)iq}$ have the form

$$h_{3iq} = A\pi^2\delta_{iq} + \iint\limits_{\tilde{S}\setminus\tilde{S}_i}\left[\frac{1}{\kappa^2}\theta_q(\boldsymbol{\eta})\tilde{R}_3(\mathbf{y}_i,\boldsymbol{\eta}) - \frac{A\delta_{iq}}{R(\mathbf{y}_i,\boldsymbol{\eta})}\right]\sin\eta_1 dS_\eta$$
$$h_{jiq} = \pi^2\delta_{iq}\left(A - \frac{1}{2}B\right) + \iint\limits_{\tilde{S}\setminus\tilde{S}_i}\left[\frac{1}{\kappa^2}\theta_q(\boldsymbol{\eta})\tilde{R}_j(\mathbf{y}_i,\boldsymbol{\eta}) - \frac{A\delta_{iq}}{R(\mathbf{y}_i,\boldsymbol{\eta})} + B\delta_{iq}\Phi_j(\mathbf{y}_i,\boldsymbol{\eta})\right]\sin\eta_1 dS_\eta$$
$$h_{j(3-j)iq} = \iint\limits_{\tilde{S}\setminus\tilde{S}_i}\left[\frac{1}{\kappa^2}\theta_q(\boldsymbol{\eta})\tilde{R}_{j(3-j)}(\mathbf{y}_i,\boldsymbol{\eta}) + B\delta_{iq}\Psi(\mathbf{y}_i,\boldsymbol{\eta})\right]\sin\eta_1 dS_\eta; \quad j=1,2 \tag{2.9}$$

After determining the nodal values of the functions $\tilde{\alpha}_j$ (j = 1, 2, 3) from systems (2.8) we can easily obtain all the most important parameters of the oscillatory process concerning both the stress–strain state of the body (relations (1.9), (2.1) and (2.6)), and the motion of the inclusion as a rigid whole (relations (1.10), (2.1) and (2.6)). The stress intensity factors of the break K_1, of the transverse shear K_2 and the longitudinal shear K_3 in the neighbourhood of the inclusion are of particular interest. These are given, as functions of the angular coordinate φ of the point of its contour, by the formulae

$$K_1(\varphi) = -2\pi\sqrt{\pi/a}\tilde{\alpha}_3(\hat{\mathbf{y}}), \quad K_2(\varphi) = -2\pi\sqrt{\pi/a}[\tilde{\alpha}_1(\hat{\mathbf{y}})\cos\varphi + \tilde{\alpha}_2(\hat{\mathbf{y}})\sin\varphi] \tag{2.10}$$

$$K_3(\varphi) = -2\pi\sqrt{\pi/a}[\tilde{\alpha}_1(\hat{\mathbf{y}})\sin\varphi - \tilde{\alpha}_2(\hat{\mathbf{y}})\cos\varphi]; \quad \hat{\mathbf{y}} = \mathbf{y}(\pi/2, \varphi)$$

3. NUMERICAL RESULTS

Consider the response of a body with a disc-shaped absolutely rigid inclusion to the propagation of a plane longitudinal displacement wave

$$\mathbf{u}^{in}(\mathbf{x}^*) = \mathbf{e}U_0 \exp[i\omega_1(\mathbf{e}\cdot\mathbf{x}^*)] \tag{3.1}$$

where U_0 is the constant amplitude of the oscillations, $\mathbf{e} = \mathbf{e}(\sin\psi, 0, \cos\psi)$ and ψ is the angle of incidence of the wave on the inclusion.

If the exciting wave propagates along the Ox_3 axis, that is, it has a wave front parallel to the inclusion (ψ = 0), the components of its displacements in the region S take the form

$$u_1^{in} = u_2^{in} = 0, \quad u_3^{in}(\mathbf{x}) = U_0$$

If the exciting wave propagates along the Ox_1 axis, i.e. it has a wave front perpendicular to the inclusion (ψ = π/2), the corresponding components take the form

$$u_1^{in}(\mathbf{x}) = U_0 \exp(i\omega_1 x_1), \quad u_2^{in} = u_3^{in} = 0$$

In the calculations we used discretization of the region S into 176 elements (11 elements along the Oy_1 axis and 16 elements along the Oy_2 axis), and also a piecewise-constant approximation of the required functions, specified by the relations

$$\theta_q(\mathbf{y}) = \begin{cases} 1, & \text{if } \mathbf{y} \in \tilde{S}_q \\ 0, & \text{if } \mathbf{y} \notin \tilde{S}_q \end{cases} \tag{3.2}$$

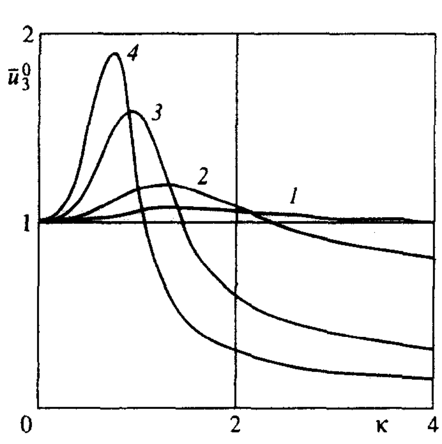

Fig. 1

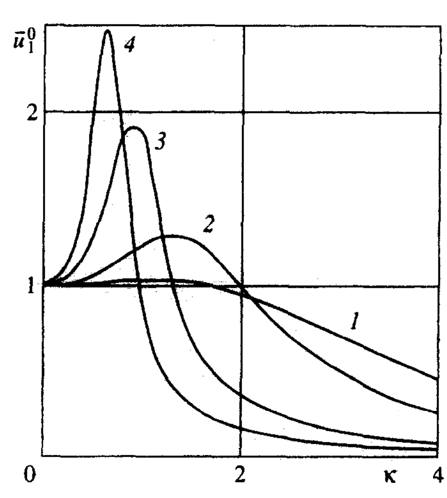

Fig. 2

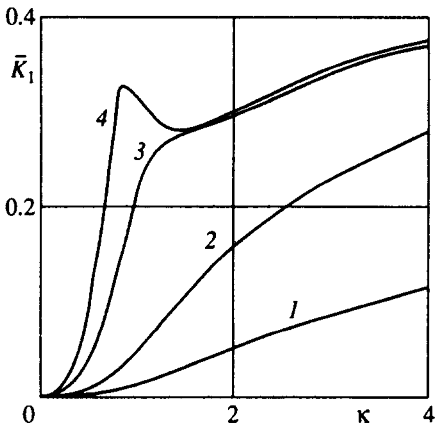

Fig. 3

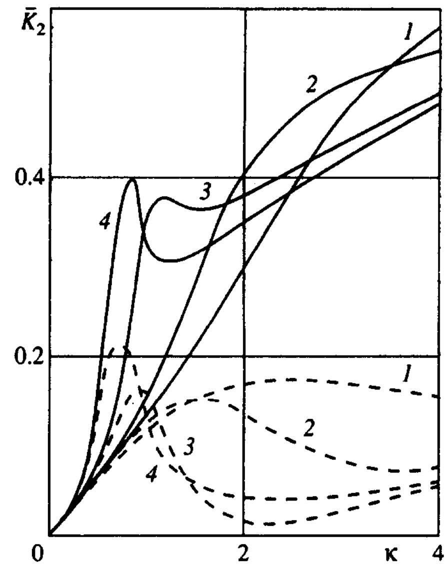

Fig. 4

Poisson's ratio was assumed to be equal to 0.3, and for the configuration of the inclusion considered $i_1 = i_2 = a/2, i_3 = a/\sqrt{2}$.

In Figs 1 and 2 we show graphs of the relative amplitudes of the translational displacements of the inclusion as a rigid whole $\bar{u}_j^0 = |u_j^0|/U_0$ against the normalized wave number $\kappa = \omega_2 a$. Figures 3–5 show similar graphs of the relative amplitudes of the stress intensity factor $\bar{K}_j = |K_j|/K_*$, where $K_* = 2\pi\sqrt{\pi/a}U_0G$. Curves 1–4 correspond to the following values of the reduced masses of the inclusion

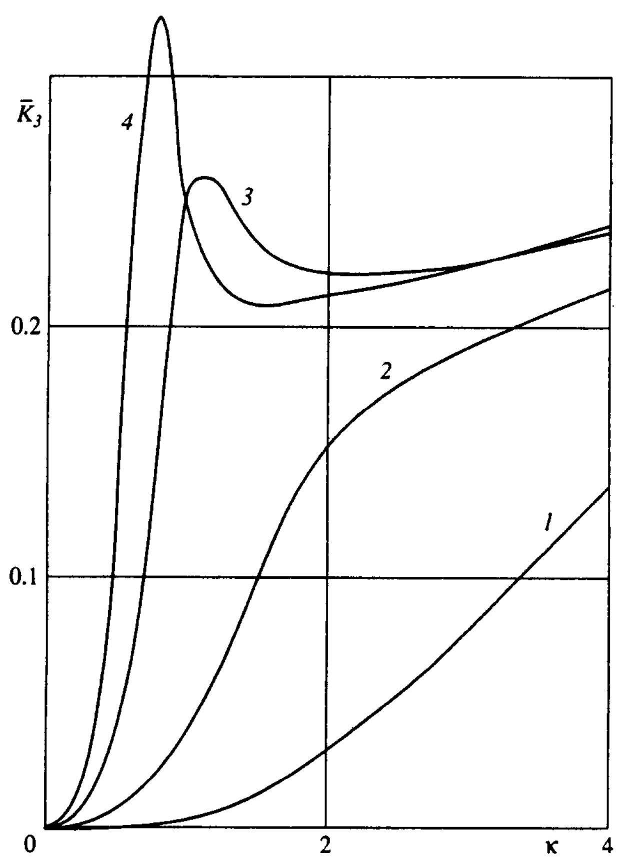

Fig. 5

$\bar{M} = M/(\rho a^3)$ (ρ is the density of the matrix): 1, 3, 10 and 20. Figures 1 and 3 correspond to the case when $\psi = 0$, in which case $u_1^0 = u_2^0 = 0$, $\Omega_1 = \Omega_2 = \Omega_3 = 0$, $K_2 = K_3 = 0$, and the stress intensity factor of the break K_1 does not vary along the contour of the defect. Figures 2, 4 and 5 correspond to the case when $\psi = \pi/2$, in which case $u_2^0 = u_3^0 = 0$, $\Omega_1 = \Omega_2 = \Omega_3 = 0$, $K_1 = 0$, and the shear stress intensity factors K_2 and K_3 depend on the angular coordinate φ of the point of the contour of the defect, measured from the Ox_1 axis. Hence, in Fig. 4 we show $\bar{K}_2$ at the most representative points of the contour of the inclusion: the continuous curves correspond to the point where the wave encounters the defect ($\varphi = \pi$), and the dashed curves correspond to the point where the wave leaves the defect ($\varphi = 0$). Since $K_3(0) = K_3(\pi) = 0$, in Fig. 5 we show values of K_3 at an intermediate point of the contour with coordinate $\varphi = \pi/2$.

It follows from Fig. 1 that for normal incidence of the wave on the inclusion, as the wave number increases the amplitude of the displacements $\bar{u}_3^0$ increases from unity when $\kappa = 0$ (the static value) to an absolute maximum, and then decreases monotonically to zero. This behaviour, which is more pronounced for inclusions of large mass, shifts into the region of lower wave numbers when the absolute maximum of $\bar{u}_3^0$ increases. In the initial range of wave numbers the amplitude of the stress intensity factor $\bar{K}_1$ gradually increases from a zero value (Fig. 3). A further increase in κ in the case of an inclusion of large mass leads to a local maximum of $\bar{K}_1$. For higher wave numbers, a characteristic feature is the convergence of the values $\bar{K}_1$ for inclusions of different mass, subsequently reaching a linear relationship between $\bar{K}_1$ and κ, which agrees with the power increase in the stresses in the generating wave.

A similar behaviour (with a difference in the quantitative factors) is observed for tangential incidence of the wave on the inclusion, for amplitudes of the longitudinal displacements $\bar{u}_1^0$ (Fig. 2), the amplitudes of the transverse-shear stress intensity factor $\bar{K}_2$ at the point of incidence of the wave (Fig. 4) and the amplitudes of the longitudinal-shear stress intensity factor $\bar{K}_3$ (Fig. 5). Absolute maxima of $\bar{K}_2$ are recorded at the point where the wave descends (Fig. 4), where $\bar{K}_2(\pi) > \bar{K}_2(0)$. There is a difference in the nature of the convergence of the stress intensity factors in the region of high wave numbers: whereas for K_3 (as also for K_1 in the previous case) as the mass of the inclusion increases this convergence is from above, for K_2 it is from below.

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